GIRRAWEEN HIGH SCHOOL

TRIAL EXAMINATION

2018

MATHEMATICS

Time allowed - Three hours
(Plus 5 minutes reading time)

General Instructions

- Reading time-5 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11-15, show relevant mathematical reasoning and/or calculations.

Total marks : 100

Part A- 10 marks

- Attempt Questions1-10
- Allow about 15 minutes for this section

Part B - 90 marks

- Attempt Questions 11-15
- Allow about 2 hours and 45 minutes for this section

PART A: Questions 1 – 10 (Multiple Choice) ***Write the letter corresponding to the correct answer on your working pages.***

Question 1

The condition for the quadratic equation $3x^2 - 12x + k = 0$ to have real roots is

A) $k \le 36$ B) $k \ge 36$ C) $k \le 12$ D) $k \ge 12$

Question 2

A yacht sailed directly from A to B on a bearing of $196^0 T$. To sail from B back to A, the bearing should be:

A) $016^0 T$ B) $074^0 T$ C) $164^0 T$ D) $196^0 T$

Question 3

$\log_3 15 + \log_3 18 - \log_3 10$ evaluates to :

A) 1 B) 2 C) 3 D) 0

Question 4

What is the size of the angle subtended at the centre of a circle with diameter 7cm by an arc 3cm long?

A) 25^0 B) 49^0 C) 67^0 D) 133^0

Question 5

What is the limiting sum of the series $45 - 15 + 5 - \frac{5}{3} + \ldots\ldots..?$

A) $\frac{135}{2}$ B) $\frac{45}{4}$ C) $\frac{135}{4}$ D) $\frac{-45}{2}$

Question 6

The area between the curve $y = \frac{1}{x}$, the x axis and the lines $x = 1$ and $x = b$ is equal to 2 square units. The value of b is

A) e B) e^2 C) $2e$ D) 3

Question 7

What is the period of the function $y = 4\sin\left(\frac{x}{3}\right)$?

A) 6π B) 4 c) $\frac{2\pi}{3}$ D) $\frac{1}{4}$

Question 8.

The first term of an arithmetic progression is 3, and eleventh term is 23. The n^{th} term is

A) $T_n = 3 + 23(n-1)$ B). $T_n = 23 + 10(n-1)$ C). $T_n = 3 + 2(n-1)$ D) $T_n = 3 + 10(n-1)$

Question 9.

If $\log_{10} 7 = a$ then $\log_{10}\left(\frac{1}{70}\right)$ is equal to

A) $-(1+a)$ B) $(1+a)^{-1}$ C) $\frac{a}{10}$ D) $\frac{1}{10a}$

Question 10.

The roots of the quadratic equation $gx^2 - x + h = 0$ are -1 and 3. The value of h is

A) -6 B) -3 C) $\frac{-3}{2}$ D) 2

PART B: Show all necessary working on your answer pages.

Question 11(18 Marks). **Marks**

a) Find the value of $e^{2\pi}$ correct to 2 significant figures. **2**

b) Factorise $3x^2 + x - 2$. **2**

c) If $\dfrac{20\sqrt{18}}{16\sqrt{24}} = a\sqrt{3}$ find the value of a. **2**

d) Solve $2\log_3 5 - \log_3 x = 2$ **2**

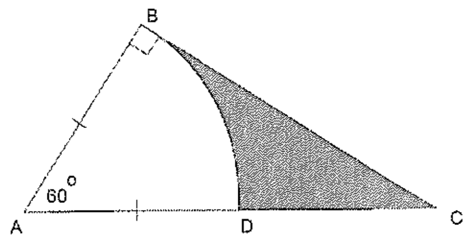
e) Given that $\tan\theta = \dfrac{7}{8}$ and $\cos\theta < 0$, find the exact value of $\operatorname{cosec}\theta$. **2**

f) Differentiate:

i) $y = \left(e^x + 2\right)^8$ **2**

ii) $y = \dfrac{x-1}{\cos x}$ **3**

g) In the diagram $\angle B = 90^0$, $\angle A = 60^0$ and $AB = AD = 10m$. BD is an arc of the circle with centre A.



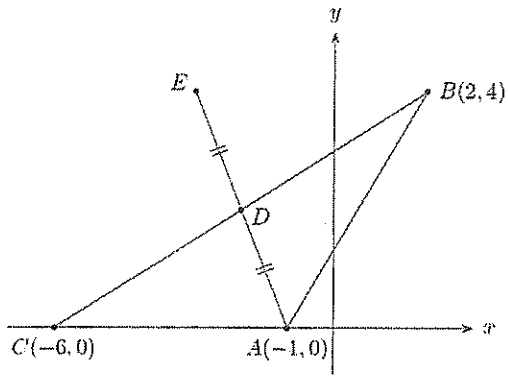
3

Calculate the shaded area in exact form.

Question 12 (18 Marks).

a) In the diagram below, A, B, C and D are the points (-1,0), (2,4) , (-6,0) (-2,2) respectively.

D is also the midpoint of AE.



In the diagram above,

i) Find the length of the interval AB. 1

ii) Find the equation of the circle with centre at B which passes through the point A. 1

iii) Find the size of $\angle CAB$ to the nearest degree. 2

iv) Find the midpoint of BC. 1

v) Show that the equation of the line BC is $x-2y+6=0$ 1

vi) Find the perpendicular distance of A from the line BC in simplest exact form. 2

vii) What type of quadrilateral is $ABEC$? Give reasons for your answer. 2

b) The roots of the equation $x^2-8x+5=0$ are α and β.

Find the value of $(\alpha-\beta)^2$. 2

c) The rate at which water flows into a tank is given by

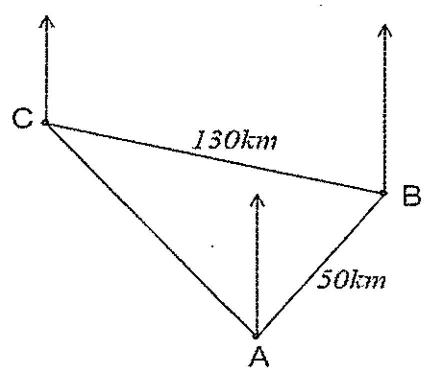
$$\frac{dV}{dt}=\frac{2t}{1+t^2},$$

Where V is the amount of water in the tank in litres and t is the time in seconds. Initially the tank is empty. Find the exact amount of water in the tank after 10 seconds. 3

d) A geometric series has first term a and limiting sum 2. Find all possible values for a. 3

Question 13 (18 Marks).

a) A ship sails 50km from port A to port B on a bearing of 063^0. It then sails 130km from port B to port C on a bearing of 296^0.



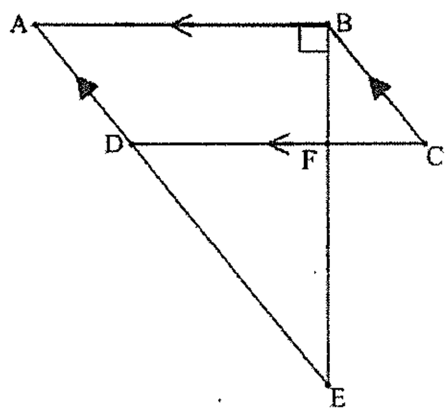
i) Show (with working) $\angle ABC = 53^0$. **2**

ii) Find AC to the nearest kilometre. **3**

b) In the following diagram, $ABCD$ is aparallelogram where $FB \perp AB$. **3**

i) Prove that $\Delta CBF \,|||\, \Delta AEB$

ii) If $CF = 3cm$, $BC = 7cm$ and $AE = 15cm$, find the length of AB. **2**



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c) Find the equation of the normal to the curve $y = 1 + \ln 2x$ at the point $\left(\frac{e}{2}, 2\right)$. **3**

d) Solve $3\sin^2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$ (Give your answer to 2 decimal places in radians) **2**

e) By letting $m = t^{\frac{1}{3}}$, or otherwise, solve $t^{\frac{2}{3}} + t^{\frac{1}{3}} - 6 = 0$ **3**

Question 14 (18 marks).

a) Consider the function $f(x)=x^3-x^2-5x+1$.

i) Find the coordinates of the stationary points of the curve $y=f(x)$ and determine their nature. **2**

ii) Find any points of inflexion. **1**

iii) Sketch the curve $y=f(x)$ for $-2\leq x\leq 2$ clearly indicating the end points.

(You need not find the x-intercepets. **2**

iv) For what values of x is the crve $y=f(x)$ decreasing but concave up? **1**

b) i) If $y=\sqrt{\sin x}$, complete the table below.

x	0	0.5	1	1.5	2
y					0.95

1

ii) Hence evaluate $\int_0^2 \sqrt{\sin x}\ dx$ using Simpson's rule with 5 function values. **2**

c) Find

α) $\int 6e^{\frac{x}{2}}dx$ **1**

β) $\int \frac{x}{1-x^2}dx$ **1**

Υ) $\int_0^{\frac{\pi}{6}}\left(1-\sec^2 2x\right)dx$ **2**

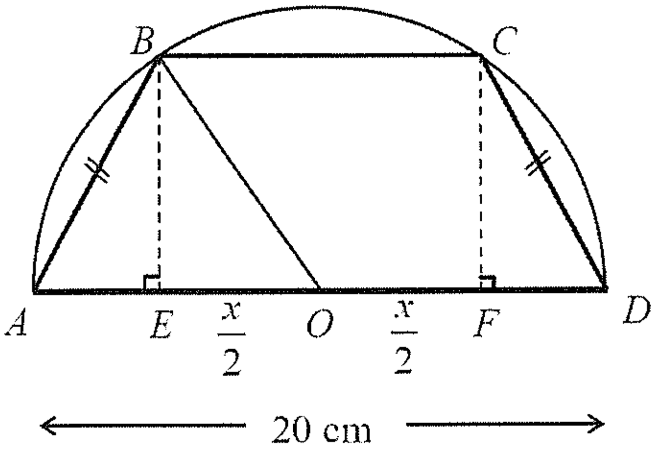
d) R is the region bounded by the y-axis, the x-axis, the line $x=\frac{\pi}{2}$ and the curve $y=\sqrt{1+\sin x}$. Find the volume of the solid formed when the region R is rotated about the x-axis. **3**

e) Consider the function $f(x)=kx^2-(3k-4)x+k$.

Show that $f(x)$ is positive definite for $\frac{4}{5}<k<4$. **2**

Question 15 (18 marks).

a) An isosceles trapezium $ABCD$ is drawn with its vertices on a semicircle centre O and diameter 20 cm. Perpendiculars BE and CF are drawn to meet the diameter AD as illustrated in the diagram below.



i) If $EO = OF = \frac{x}{2}$, show that $BE = \frac{1}{2}\sqrt{400 - x^2}$ **2**

ii) Show that the area of the trapezium $ABCD$ is given by $\frac{1}{4}(x+20)\sqrt{400-x^2}$. **2**

iii) Hence find the length of BC so that the area of the trapezium $ABCD$ is a maximum. **3**

b) A particle P is at the origin at time $t = 0$ and moves so that its velocity for $t \geq 0$ is given by $v = \frac{1}{t+1}$.

i) What is the acceleration of P when $t = 1$? **2**

ii) What is the displacement of x of P from the origin when $t = 1$? **2**

Question 15 continues on the next page

c) The population P of a town is growing at a rate proportional to its size at any time, so that $\frac{dP}{dt} = kP$, for some constant k. At the beginning of 2010 (where t=0 and time is in years)the town's population was 23000 and at the beginning of 2016 its population had grown to 28000.

i) Show that $P = Ae^{kt}$ satisfies the equation $\frac{dP}{dt} = kP$. **1**

ii) Find the value of A. **1**

iii) Find the value of k. **2**

iv) Estimate, to the nearest hundred, what the population will be at the beginning of 2025. **1**

v) During which year will the population be double the size it was at the beginning of 2010. **2**

End of examination!!!

Part A

(1) C (2) A (3) C (4) B (5) C

(6) B (7) A (8) C (9) A (10) C

Question 11

a) $e^{2\pi} = 535.49 = 540$ to 2 sig. figures

b) $3x^2 + x - 2 = (3x-2)(x+1)$

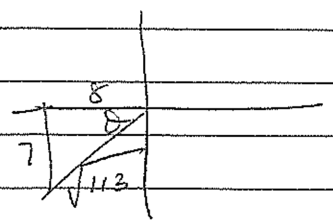
c) $\dfrac{20\sqrt{18}}{16\sqrt{24}} = \dfrac{5\sqrt{3}}{4\sqrt{4}} = \dfrac{5\sqrt{3}}{8}$ $\therefore a = \dfrac{5}{8}$

d) $2\log_3 5 - \log_3 x = 2$

$\log_3 5^2 - \log_3 x = 2$

$\log_3 \dfrac{25}{x} = 2$ $\therefore \dfrac{25}{x} = 3^2 = 9$

$\therefore x = \dfrac{25}{9}$

e) 

$\tan\theta = +ve$, $\cos\theta < 0$

$\therefore$ 3rd quadrant

$\therefore \sin\theta = \dfrac{-7}{\sqrt{113}}$

$\therefore \operatorname{cosec}\theta = -\dfrac{\sqrt{113}}{7}$

f) (i) $y = (e^x + 2)^8$

$\dfrac{dy}{dx} = 8(e^x+2)^7 \times e^x = 8e^x(e^x+2)^7$

(f)(ii) $y = \frac{x-1}{\cos x}$ $u = x-1$, $u' = 1$ | $v = \cos x$, $v' = -\sin x$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{\cos x + (x-1)\sin x}{\cos^2 x}$$

$$= \frac{\cos x + x\sin x - \sin x}{\cos^2 x}$$

g) In ΔABC; $\tan 60^\circ = \frac{BC}{10} = \sqrt{3}$ $\therefore BC = 10\sqrt{3}$

Area of $\Delta ABC = \frac{1}{2} \times 10 \times 10\sqrt{3} = 50\sqrt{3}\, m^2$

Area of sector $ABD = \frac{1}{2} \times 10 \times 10 \times \frac{\pi}{3} = \frac{50\pi}{3} m^2$

$\therefore$ shaded area $= 50\sqrt{3} - \frac{50\pi}{3}$

$$= \frac{50}{3}\left[3\sqrt{3} - \pi\right] m^2$$

Question 12

a) (i) $AB = \sqrt{(2+1)^2 + (4-0)^2} = 5$

(ii) $(x-2)^2 + (y-4)^2 = 25$

(iii) $m_{AB} = \tan\theta = \frac{4}{3}$ $\therefore \theta = \tan^{-1}\frac{4}{3} = 53.13^\circ$

$\therefore \angle CAB = 180^\circ - 53.13^\circ \doteqdot 127^\circ$

(iv) $M = \left(\frac{2-6}{2}, \frac{4-0}{2}\right) = (-2, 2)$

(v) $m_{BC} = \frac{4-0}{2+6} = \frac{1}{2}$

$B(-6, 0)$ $\therefore$ equation of $BC \Rightarrow$

$y - 0 = \frac{1}{2}(x+6) \Rightarrow x - 2y + 6 = 0$

(vi) $(-1, 0)$ $x - 2y + 6 = 0$

$\therefore$ perpendicular distance $= \left|\frac{-1 - 2(0) + 6}{\sqrt{1+4}}\right|$

$$= \frac{5}{\sqrt{5}} = \sqrt{5}$$

(vii) As D is the mid point of AE as well, $ABEC$ is a parallelogram.
(diagonals bisect each other)

But as $AC = 5$ as well as $AB = 5$, $ABEC$ is a rhombus.
(adjacent sides are equal and has properties of a parallelogram)

b) $x^2 - 8x + 5 = 0$

$\alpha + \beta = 8$ and $\alpha\beta = 5$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$= 64 - 20 = 44$

c) $\frac{dv}{dt} = \frac{2t}{1+t^2}$ when $t = 0$, $v = 0$

$\therefore v = \int \frac{2t}{1+t^2}\, dt = \ln[1+t^2] + c$

when $t = 0$, $v = 0 \Rightarrow 0 = \ln 1 + c \Rightarrow c = 0$

$\therefore v = \ln(1+100) = \ln(101)$

d) $S_\infty = \frac{a}{1-r} = 2$

$\frac{a}{2} = 1 - r \Rightarrow r = 1 - \frac{a}{2}$

For S_∞, $|r| < 1$ $\left|1 - \frac{a}{2}\right| < 1$

ie $-1 < 1 - \frac{a}{2} < 1$

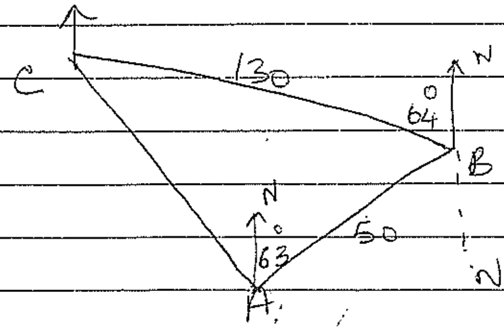
$-2 < 2 - a < 2$

$2 > a - 2 > -2 \Rightarrow 4 > a > 0$

$\therefore 0 < a < 4$

Question 13

a) (i)



$\angle ABN' = 63°$ (alternate ∠s on || lines)

$\therefore \angle ABC = 180° - (63 + 64°)$ ~~(∠s about a point = 360°)~~ (∠'s on a straight line)

$= 53°$

(ii) $AC^2 = AB^2 + BC^2 - 2AB.BC\cos 53°$

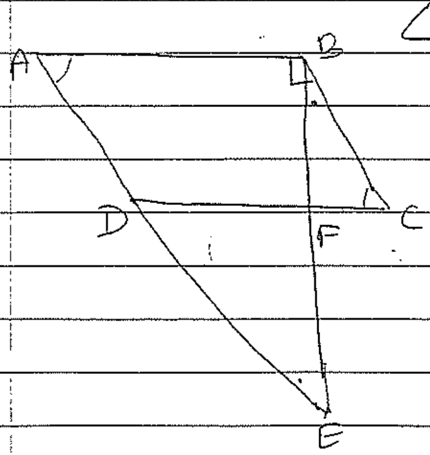
$= 50^2 + 130^2 - 2 \times 50 \times 130 \cos 53$

$= 11576$

$\therefore AC = \sqrt{11576} = 108$ km (nearest km)

(b) (i) In Δ's CBF, AEB

$\angle CBF = \angle BEA$ (alternate angles on || lines, AE||BC)



$\angle BCF = \angle EAB$ (opposite angles of a parallelogram)

$\therefore \triangle CBF ||| \triangle AEB$ (equiangular)

(ii) $\frac{AB}{CF} = \frac{AE}{BC}$ $\therefore AB = \frac{AE \times CF}{BC}$

(Ratio of matching sides) $= \frac{15 \times 3}{7} = 6\frac{3}{7}$ m c)

$$y = 1 + \ln 2x \qquad \text{at } \left(\frac{e}{2}, 2\right)$$

$$y' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$$

$$\text{at } \left(\frac{e}{2}, 2\right),\ y' = \frac{2}{e}$$

$$\therefore y'_{\text{normal}} = -\frac{e}{2}$$

$\therefore$ equation of the normal:

$$y - 2 = -\frac{e}{2}\left(x - \frac{e}{2}\right)$$

$$\cancel{2y - 4 = -ex + \frac{e^2}{2}} \Rightarrow \cancel{4y - 8 = -2ex + e^2}$$

$$\therefore 2ex + 4y - 8 - e^2 = 0$$

d)

$$3\sin^2\theta = \sin\theta \qquad \text{for } 0 \le \theta \le 2\pi$$

$$3\sin^2\theta - \sin\theta = 0$$

$$\sin\theta(3\sin\theta - 1) = 0$$

$$\therefore \sin\theta = 0 \quad \underline{\text{or}} \quad 3\sin\theta = 1$$

$$\sin\theta = \frac{1}{3}$$

$$\therefore \theta = 0, \pi, 2\pi \quad \underline{\text{or}} \quad \theta = 0.34, 2.8$$

$$\therefore \theta = 0, 0.34, 2.8, \pi, 2\pi$$

e)

$$m = t^{1/3} \qquad \therefore m^2 = (t^{1/3})^2 = t^{2/3}$$

$$t^{2/3} + t^{1/3} - 6 = 0$$

$$\text{ie} \quad m^2 + m - 6 = 0$$

$$(m+3)(m-2) = 0$$

$$\therefore m = -3 \quad \text{or} \quad m = 2$$

$$t^{1/3} = -3 \quad \underline{\text{or}} \quad t^{1/3} = 2$$

$$\therefore t = -27 \qquad \therefore t = 8$$

Question 14

a) $f(x) = x^3 - x^2 - 5x + 1$

$f'(x) = 3x^2 - 2x - 5$

$f''(x) = 6x - 2$

$f'(x) = 0$ for stationary points

ie $3x^2 - 2x - 5 = 0$

$(3x-5)(x+1) = 0$

$\therefore x = -1$ or $\frac{5}{3}$

If $x = -1$, $f''(-1) = 6(-1) - 2 < 0$ $\therefore$ maximum

$f(-1) = (-1)^3 - (-1)^2 - 5(-1) + 1$

$= -1 - 1 + 5 + 1 = 4$

$\therefore$ a maximum at $(-1, 4)$

If $x = \frac{5}{3}$, $f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) - 2 > 0$ $\therefore$ minimum

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) + 1$

$= -5\frac{13}{27}$

$\therefore$ a minimum at $\left(\frac{5}{3}, -5\frac{13}{27}\right)$

(ii) $\therefore$ ~~for~~ Points of inflexion when $f''(x) = 0$

ie $6x - 2 = 0$ $\therefore x = \frac{1}{3}$

x	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
$f''(x)$	-0.5	0	1

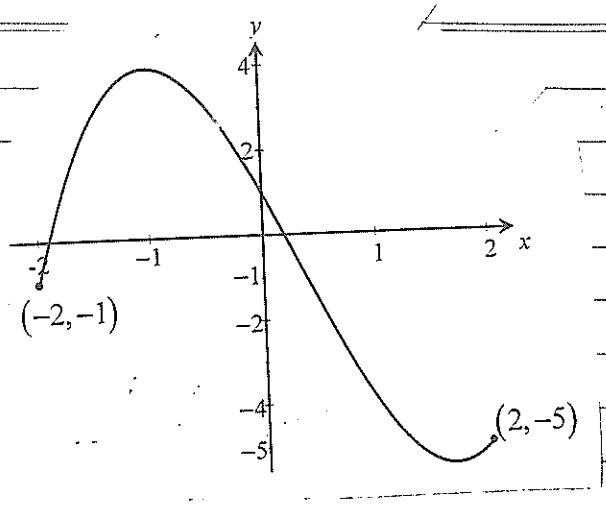
$\therefore$ concavity changes

$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 5\left(\frac{1}{3}\right) + 1$

$= -\frac{20}{27}$

$\therefore$ an inflexion at $\left(\frac{1}{3}, \frac{-20}{27}\right)$

(iii)



(iv)

$y = f(x)$ is decreasing, but concave up when

$$\frac{1}{3} < x < \frac{5}{3}$$

b) (i) $y = \sqrt{\sin x}$

x	0	0.5	1	1.5	2
y	0	0.69	0.92	1.0	0.95

(ii) $A = \frac{0.5}{3}\left[0 + 0.95 + 4(0.69 + 1) + 2(0.92)\right]$

$= 1.5916\ldots = 1.6$

~~c) (iii)~~ α)

$$\int 6e^{\frac{x}{2}}\,dx = 6 \cdot \frac{e^{\frac{x}{2}}}{\frac{1}{2}} = 12e^{\frac{x}{2}} + C$$

β) $\int \frac{x}{1-x^2}\,dx = -\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx = -\frac{1}{2}\log_e(1-x^2) + C$

(γ) $\int_0^{\frac{\pi}{6}} (1 - \sec^2 2x)\,dx = \left[x - \frac{\tan 2x}{2}\right]_0^{\frac{\pi}{6}}$

$= \left(\frac{\pi}{6} - \frac{1}{2} \cdot \tan\frac{\pi}{3}\right) - (0 - 0)$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Question 14 continued

d) $V = \pi \int_0^{\frac{\pi}{2}} (1 + \sin x)\, dx$

$= \pi \left[x - \cos x \right]_0^{\pi/2}$

$= \pi \left[\frac{\pi}{2} - \cos\frac{\pi}{2} - 0 + 1 \right]$

$= \pi \left[\frac{\pi}{2} + 1 \right] = \frac{\pi}{2}(\pi + 2)\ u^3$

e) $f(x) = kx^2 - (3k-4)x + k$

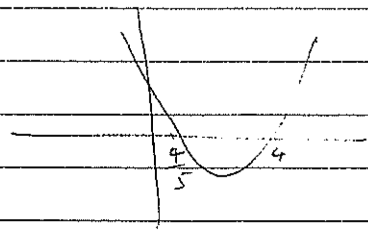
positive definite $\Rightarrow$ $\Delta < 0$ and $a > 0$.

$\Delta = (3k-4)^2 - 4k^2 = 9k^2 - 24k + 16 - 4k^2$

$= 5k^2 - 24k + 16 < 0$

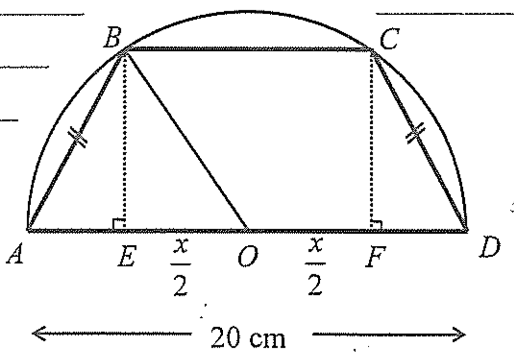
$\Rightarrow (5k-4)(k-4) < 0$

$\therefore \frac{4}{5} < k < 4$



Question 15

a)



diameter = 20cm

∴ radius = 10cm

(i) In $\triangle OBE$, $OB^2 = BE^2 + OE^2$ (Pythagoras' theorem)

$$\therefore 10^2 = BE^2 + \frac{x^2}{4} \Rightarrow BE^2 = 100 - \frac{x^2}{4}$$

$$= \frac{1}{4}(400 - x^2)$$

$$\therefore BE = \frac{1}{2}\sqrt{400 - x^2} \quad (\text{length} \geq 0)$$

(ii)

$$A_{Trap.} = \frac{1}{2}h(a+b)$$

$$= \frac{1}{2} \cdot BE\,(BC + AD)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \sqrt{400 - x^2}\,(x + 20)$$

$$= \frac{1}{4}(x+20)\sqrt{400 - x^2}$$

(iii) $A = \frac{1}{4}(x+20)\sqrt{400-x^2} = \frac{1}{4}(x+20)(400-x^2)^{\frac{1}{2}}$

$$\therefore \frac{dA}{dx} = \frac{1}{4} \cdot (x+20) \cdot \frac{1}{2}(400-x^2)^{-\frac{1}{2}}(-2x) + (400-x^2)^{\frac{1}{2}} \cdot \frac{1}{4}$$

$$= -\frac{x}{4}(x+20)(400-x^2)^{-\frac{1}{2}} + \frac{1}{4}(400-x^2)^{\frac{1}{2}}$$

$$= \frac{1}{4}\left[\sqrt{400-x^2} - \frac{x(x+20)}{\sqrt{400-x^2}}\right]$$

Maximum occurs when $\frac{dA}{dx} = 0$ $\Rightarrow \frac{1}{4}\left[\frac{400 - x^2 - x^2 - 20x}{\sqrt{400-x^2}}\right] = 0$

$$= \frac{-\frac{1}{4}(2x^2 + 20x - 400)}{\sqrt{400 - x^2}} = 0$$

$$\therefore 2x^2 + 20x - 400 = 0$$

$$x^2 + 10x - 200 = 0$$

$$(x+20)(x-10) = 0$$

$$\therefore x = 10 \quad (x > 0)$$

x	9	10	11
$\frac{dA}{dx}$	0.8	0	-0.796

$\therefore$ The maximum occurs ~~at~~ when $x = 10$

i.e. $BC = 10$ cm.

Question 15

b) (i) $V = \dfrac{1}{t+1} = \dfrac{dx}{dt}$

$$\therefore \ddot{x} = \frac{dv}{dt} = -(1+t)^{-2} = \frac{-1}{(1+t)^2} = -\frac{1}{4}$$

when $t = 1$

(ii) $$\int dx = \int \frac{dt}{1+t} = \ln(t+1) + c$$

$$\therefore x = \ln(t+1) + c$$

$$t = 0,\ x = 0 \Rightarrow \quad 0 = \ln 1 + c \Rightarrow c = 0$$

$$\therefore x = \ln(1+t) = \ln 2$$

when $t =$

Question 15 continued

c)

(i) $P = Ae^{kt}$

$\frac{dP}{dt} = k.Ae^{kt} = kP$

(ii) $P = Ae^{kt}$

In 2010, $t = 0$, $P = 23000$

$\therefore 23000 = A.e^{0} = A$

$\therefore A = 23000$

(iii) In 2016, $t = 6$, $P = 28000$

$28000 = 23000e^{6k}$

$\therefore k = \frac{1}{6}\ln\left(\frac{28}{23}\right)$

$\doteqdot 0.032785\ldots$

(iv) $t = 15$, $P = ?$

$P = 23000e^{15k}$

$\doteqdot 37600$ (to nearest hundred)

(v) $P = 46000$, $t = ?$

$46000 = 23000e^{kt}$

$\therefore kt = \ln 2$

$\therefore t = \frac{1}{k}\ln 2$

$\doteqdot 21.14$

$\therefore$ doubles during the 22^{nd} year

ie 2031